ISSN 2229-5518

# Total Zero-Divisor Graphs of Idealizations with Respect to Prime Modules

**D.Eswara Rao D.Bharathi**

**Abstract:** Let R be a commutative ring with identity and let M be a prime R-module. let R(+)M be the idealization of ring R by the R-module M. we study the diameter and girth of the Total zero divisor graph of the ring R(+)M. In this paper we discuss the Total zero divisor graphs of idealization with respect to the prime modules. In this, we consider R be a commutative ring and let M be a P-prime R- module and P = (0:M) then we prove that (i) if P≠0 then (a,m) ∈ Z(R(+)M) if and only if a ∈ P∪Z(R) (ii) if P = 0 then (a,m) ∈ Z(R(+)M) if and only if a = 0 and m ∈ M*. Using this result we prove that let Z(Γ(R) ≠ φ, Z(R) is an ideal of R then Z(Γ(R(+)M)) is complete if and only if Z(R) ⊆(0:M). and also we prove that (i) if P = 0 then Z(Γ(R(+)M)) is complete, (ii) if P≠0 and Z(R) is not an ideal of R then diam(Z(Γ(R(+)M))) = 2. Also we show that if |P|=0 then diam(Z(Γ(R(+)M))) = 1,if |P|≠0 then diam(Z(Γ(R(+)M))) = 2.

**Index Terms:** Zerodivisors, Total zerodivisor graph of idealization, commutative ring, connected graph, prime module.

## 1. Introduction:

Let R be a commutative Ring with non zero unity. The concept of the graph of the zero divisors of R was first introduced by Beck [1], where he was mainly interested in coloring. In his work all elements of the ring were vertices of the graph. The investigation of colorings of a commutative ring was then continued by D. D. Anderson and Naseer [2], In [3], D. F. Anderson and Livingston associate a graph, Γ(R), to R with vertices $Z(R)^*=Z(R)\setminus\{0\}$, the set of non zero zero divisors of R, and for distinct $x,y \in Z(R)\setminus\{0\}$. The vertices x and y are adjacent if xy=0. In [5] D.F. Anderson and Badawi introduced the total graph of R, denoted by T(Γ(R)) as the graph with all elements of R as vertices, and for distinct x, y ∈ R are adjacent if x+y ∈ Z(R), they studied some graphical parameters of this graph such as diameter and girth.

Research Scholar, Department of Mathematics, S.V.University, Tirupathi.
Email: msceswar@gmail.com
Associate Professor, Department of Mathematics, S.V.University, Tirupathi.,
Email: bharathikavali.yahoo.co.in

we study some results of Total graphs of idealizations with respect to prime module . In [5] D.F.Anderson, A.Badawi studied connectedness of Total graph of the idealization R(+)M and also investigate diameter and has proved some results on girth of Total graphs. Different aspects of the idealization are thoroughly investigated in [10],[11]. In this paper we also extend the study of D.F.Anderson, and A.Badawi with respect to prime module . In this section consider R be a commutative ring and let M be a P-prime R-module and P = (0:M) then we prove that (i) if P≠0 then (a,m) ∈ Z(R(+)M) if and only if a ∈ P∪Z(R) (ii) if P = 0 then (a,m) ∈ Z(R(+)M) if and only if a = 0 and m ∈ M*. Using this result we prove that let Z(Γ(R) ≠ ϕ, Z(R) is an ideal of R then Z(Γ(R(+)M)) is complete if and only if Z(R) ⊆(0:M). and also we prove that (i) if P = 0 then Z(Γ(R(+)M)) is complete, (ii) if P≠0 and Z(R) is not an ideal of R then diam(Z(Γ(R(+)M))) = 2. Also we show that if |P|=0 then diam(Z(Γ(R(+)M))) = 1,if |P|≠0 then diam(Z(Γ(R(+)M))) = 2.

## 2. Preliminaries:

ISSN 2229-5518

**Complete Graph:** A graph G in which every vertex is adjacent to every other vertex is called a complete graph. Complete graph is represented as $K_n$ where n is the number of vertices in $K_n$ .

**Connected Graph:** A graph G is said to be a connected graph if there is at least one path between every pair of vertices in G. otherwise G is said to be a disconnected graph.

**Distance:** Any two distinct vertices a and b in graph G , the distance between a and b, denoted by d(a,b) is the length of a shortest path connecting a and b ,if such a path exist. Otherwise d(G) = ∞

**Diameter of G :** diam(G) = Sup{d(x,y) / x & y are distinct vertices in G},where d(x,y) is the length of shortest path from x to y in G. if there is no such a path then d(x,y) = ∞.

**The girth of G:** The girth of G is denoted by gr(G) is length of shortest cycle in G. if G contains no cycles the gr(G) = ∞.

**Path:** A trail in which all the vertices are distinct is called a path.

**Cycle:** A path whose origin and terminus vertices are the same is called a cycle.

**The idealization of M over R:**
The idealization of M over R is the commutative ring formed from R×M by defining addition and multiplication as follows

(i) $(r_1,m_1)+(r_2,m_2)=(r_1+r_2,m_1+m_2)$;

(ii) $(r_1,m_1)(r_2,m_2)=(r_1r_2,r_1m_2+r_2m_1)$.

The idealization of *M* in *R*, denoted by *R*(+)*M*, We will assume that neither the ring nor the module is trivial. Observe that if $a\in Z(R)^*$, then $(a,m)\in Z(R(+)M)^*$for all $m\in M$. To see this, consider $b\in Z(R)^*$ with $ab$=0. If $bM$=0, then $(a,m)(b,0)$=0. If $bM\neq0$, then there exists some $n\in M$ such that $bn\neq0$. Hence, $(a,m)(0,bn)$=0.

## 3. Main Results:

Let M be a P-prime module over a Commutative ring R. $V_1$= {(0, m) : m ∈ M*}, $V_2$= {(a, n) : a ∈ P*, n ∈M} and $V_3$= {(a, n) : a ∈ Z*(R), n ∈ M} are used in this section.

**Theorem4.2.1** Let R be a commutative ring and let M be a P-prime R-module. Then

(i) If P ≠ 0, then (a, m) ∈ Z(R(+)M) if and only if a ∈ P ∪Z(R).

(ii) If P = 0, then (a, m) ∈ Z(R(+)M) if and only if a = 0 and m ∈ M*.

**Proof.** (i) Let (a, m) ∈ Z(R(+)M)). We may assume that a ≠ 0.

There exist a non-zero element (b, n) of R(+)M such that (a, m)(b, n)=(ab, an + bm) =(0,0).

If b = 0, then a ∈ (0 : n) = P; if b ≠ 0, then a ∈ Z(R).

Conversely, assume that (a, m) ∈ R(+)M with a ∈ P ∪Z(R).

If a ∈ Z(R), then ab = 0

for some non-zero element b ∈ R. If b ∈ P, then (a, m)(b,0) = (0,0). If b ∉ P,

then there is an element x of M such that bx ≠ 0.

Then (a, m)(0, bx) = (0,0).

Finally, if a ∈ P, then there exists a non-zero element y of M such that ay = 0.

Therefore, (a, m)(0, y) = (0,0), and so the case.

(ii) Let (a, m) ∈ Z(R(+)M)). We may assume that a ≠ 0.

There exist a non-zero element (b, n) of R(+)M such that ab = 0 and an+bm = 0.

Since M is a 0-prime R-module, we must have R is an integral domain;

hence if a ≠ 0, then b = 0, n ≠ 0 and a ∈ (0 : n) = 0 which is a contradiction.



Therefore, a = 0 and m ≠ 0 since (a, m) ≠ 0.

The other implication is clear. ♦

**Theorem4.2.2** Let R be a commutative ring and let M be a P-prime R-module. Then

(i) If P = 0, then Z(R(+)M)*= $V_1$.

(ii) If P ≠ 0 and Z(R)*≠ ∅, then Z(R(+)M)*= $V_1$∪$V_2$∪$V_3$.

(iii) If P ≠ 0 and Z(R)*= ∅, then Z(R(+)M)*= $V_1$∪$V_2$.

**Proof.** This follows from theorem 4.2.1.

**Theorem 4.2.3** Let M be a prime module over a commutative ring R and let Z(Γ(R)) ≠ ∅. Then Z(Γ(R(+)M)) is complete if and only if Z(R) ⊆ (0 :M).

**Proof.** Since Z(Γ(R)) ≠ ∅, we must have (0 : M) = P ≠ 0.

Assume Z(Γ(R(+)M)) is complete.

Let r ∈ Z(R), 0 ≠ m ∈ M. We may assume that r ≠ 0. Then

Theorem 4.2.2 gives (0, m),(r,0) ∈ Z(R(+)M)*

Hence $(0, m)+(r,0) \in Z(R(+)M)^* \Rightarrow (r,m) \in Z(R(+)M)^*$ [ by the hyp.]
Then for some $(b,n) \in R(+)M$ we have $(r,m).(b,n) = 0 \Rightarrow (rb, rn+bm) = 0$
$\Rightarrow$
$rb = 0$ and $rn+bm = 0$
If $b = 0 \Rightarrow rn = 0 \Rightarrow r \in (0:M)$
$\Rightarrow$
$Z(R) \subseteq (0:M)$
If $b \neq 0 \Rightarrow$ and $rb = 0$ ,b is a zero divisor, i.e. $b \in Z(R)$.
For some $m \in M \subseteq R$ we have $bm = 0$ this gives $rn = 0 \Rightarrow r \in (0:M) \Rightarrow$ $Z(R) \subseteq (0:M)$.
Conversely assume that $Z(R) \subseteq (0:M) = P$.
Let $(a,m),(b,n) \in Z(R(+)M)^*$ and $(a,m)+(b,n) = (a+b,m+n)$
For some $(c,l) \in R(+)M$ we have $(a+b,m+n).(c,l) = (ac+bc , al+bl+mc+nc)$
If $a = b = c = 0$ then clearly $(a+b,m+n).(c,l) = 0$.
If $c = 0$ and $a,b \in Z(R)^* \subseteq P$ then $al = 0, bl = 0$. hence $(a+b,m+n).(c,l) = 0$.
If $a,b,c \in Z(R)^* \subseteq P$ then $ac = 0, bc = 0$, and $al = 0, bl = 0, mc = 0, nc = 0$ then
$(a+b,m+n).(c,l) = 0$.
So for all cases $(a,m)+(b,n) = (a+b,m+n) \in Z(R(+)M)^*$
Thus $Z(\Gamma(R(+)M))$ is complete. ♦

Note that if M is a prime R-module, then any non-zero sub module of M is prime. Therefore, by Theorem 4.2.3, we have the following corollary:

**Corollary4. 2.4** Let R be a commutative ring, M a prime R-module, N a nonzero sub module of M and $Z(\Gamma(R)) \neq \emptyset$. Then $Z(\Gamma(R(+)M))$ is complete if and only if $Z(\Gamma(R(+)N))$ is complete.

**Theorem4.2.5** : Let M be a P-prime module over a commutative ring R and let $Z(\Gamma(R)) = \emptyset$. Then:
(i) If $P = 0$, then $Z(\Gamma(R(+)M))$ is complete.
(ii) If $P \neq 0$, then $\text{diam}(Z((\Gamma(R(+)M))) = 2$.

**Proof :** (i) Since $Z(\Gamma(R)) = \emptyset$, we must have R is an integral domain. If $P = 0$,
then Theorem 4.2.2 gives $Z(R(+)M)^* = V_1$, so clearly it is complete.
(ii) If $P \neq 0$,and $Z(R)$ is not an ideal of R then Theorem 4.2.2 gives $Z(R(+)M)^* = V_1 \cup V_2$.
Let $z_1=(a, m)$, $z_2= (b, n) \in Z(R(+)M)^*$. If $z_1, z_2 \in V_1$, then $z_1 + z_2 \in Z(R(+)M)^*$.
If $z_1 \in V_2$ and $z_2 \in V_1$, then $a \in P$ and $b = 0$; $z_1 + z_2 = (a+b,m+n)$ for some $(c,l) \in R(+)M$
we have $(a+b,m+n).(c,l) = (ac+bc , al+bl+mc+nc)$
if $c = 0$ then $(a+b,m+n).(c,l) = 0$ so $z_1 + z_2 = (a+b,m+n) \in Z(R(+)M)^*$.
If $c \neq 0$ and $a \in Z(R)$ then $ac = 0$ for some $c \in P \subseteq R$
Since $c \in P$ then $mc+nc = 0$ so $z_1 + z_2 = (a+b,m+n) \in Z(R(+)M)^*$.
If $c \notin P$ and let $m = -n$ then $(a+b,m+n).(c,l) = (ac , mc+nc) = (0,0)$ then $z_1 + z_2 \in Z(R(+)M)^*$.
Similarly, if $z_1 \in V_1$ and $z_2 \in V_2$, then $z_1 + z_2 \in Z(R(+)M)^*$.
Suppose that $z_1, z_2 \in V_2$ and let $0 \neq x \in M$ then $a,b \in P = (0:M)$
Hence $z_1 - (0,x) - z_2$ is a path.
Since $(a,m) + (0,x) \in Z(R(+)M)^*$ and $(0,x) + (b,n) \in Z(R(+)M)^*$.
Because $(a,m) + (0,x) = (a,m+x)$ for $(c,l) \in R(+)M$
$(a,m+x).(c,l) = (ac, al+mc+xc)$
If $c = 0$, $a \in P \Rightarrow (a,m+x).(c,l) = 0$
If $c \neq 0, c \in P \subseteq R, a \in P \Rightarrow mc+xc = 0, ac = 0 \Rightarrow (a,m+x).(c,l) = 0$
If $c \notin P$ and let $m = -x$ then $mc+xc = 0 \Rightarrow (a,m+x).(c,l) = 0$
Hence $(a,m) + (0,x) \in Z(R(+)M)^*$ and similarly $(0,x) + (b,n) \in Z(R(+)M)^*$.
So $z_1 - (0,x) - z_2$ is a path in $Z((\Gamma(R(+)M))$.
Hence $\text{diam}(Z((\Gamma(R(+)M))) = 2$. ♦

**Theorem 4.2.6** Let M be a P-prime module over a commutative ring R and let $Z(\Gamma(R)) = \emptyset$. Then $\text{diam}(Z((\Gamma(R(+)M)))) \leq 2$.

**Proof**. This follows from theorem 4.2.5.

**Example 4.2.7** (i) Since Z is a 0-prime Z-module, we must have $Z(\Gamma(R(+)M))$ is complete by Theorem4. 2.5 (i).
(ii) Let $M = Z_3$ denote the ring of integers modulo 3. Then M is a 3Z-prime Z-module.
Then $\text{diam}\, Z((\Gamma(R(+)M))) = 2$ by Theorem 4.2.5 (ii)

**Lemma 4.2.8** Let R be a commutative ring with identity and $M \cong Z_3$ a P-prime R-module. Then:
(i) $P \neq 0$ if and only if $|R| > 3$.
(ii) $P = 0$ if and only if $|R| = 3$.

Proof. (i) Since $P \neq 0$ and it is prime, we must have $|P| \geq 3$; hence $|R| \geq 4$.
Conversely, assume that $|R| \geq 4$, so by [2, p. 237], there always exists a
non-zero $r \in R$ such that $rZ_3 = 0$. Therefore, $P \neq 0$.

(ii) is clear. ♦

**Theorem 4.2.9.** Let R be a commutative ring and let M be a P-Prime module

(i) $|P| = 0$ $[P = \emptyset]$ then diam $Z((\Gamma(R(+)M))) = 1$ .

(ii) $|P| \geq 1$ then diam $Z((\Gamma(R(+)M))) = 2$ .

Proof : (i) if $|P| = 0$ $[P = \emptyset]$

i.e. there is no element $r \in R^*$ such that $rm = 0$ for any $m \in M^*$ .

then theorem 4.2.2 gives $Z(R(+)M)^* = V_1$

then for any two elements in $V_1$ are distance 1.

diam $Z((\Gamma(R(+)M))) = 1$.

(ii) if $|P| \geq 1$

i.e. there exist at least one element $m \in M^*$ such that $rm = 0$ for any $r \in R$.

then $Z(R(+)M)^* = V_1 \cup V_2 = \{(0,m): m \in M^*\} \cup \{(r,m), \ldots\ldots : m \in M^*\}$

then any two elements of $V_2$ are not adjacent. But $(r,n) + (0,m) \in Z(R(+)M)^*$ and $(0,m) + (0,n) \in Z(R(+)M)^*$. Then $(r,n)$ - $(0,m)$-$(0,n)$, $m \neq n$ and $m,n \in M^*$ is a path in $Z((\Gamma(R(+)M)))$.

Then diam $Z((\Gamma(R(+)M))) = 2$.

## 4. References:


[1]. I.Beck, Coloring of commutative rings, J.Algebra 116(1988),208-226.

[2]. D.D.Anderson,M.Naseer, Becks coloring of a commutative ring, J.Algebra,159(1993), 500-514.

[3].D.F. Anderson,P.S.Livingston, The zero divisor graph of a commutative ring J.Algebra, 217(1999),434-447.

[4]. D.F. Anderson, A.Frazier,A.Lauve, P.Livingston ,The zero divisor graph of a commutative ring II , Lecture Notes in Pure and Appl.Math.,vol.220,Dekker,New York,2001,61-72.

[5]. D.F. Anderson,A.Badawi, The total graph of commutative ring, J.Algebra,.320, 2706-2719,(2008)

[6]. F.R.DEMEYER,T.McKenzie and K.Schneider,the zero divisor graph of a commutative semigroup. Semigroup Forum,vol.65(2002),206-214.

[7]. D.F. Anderson, A. Badawi, On the zero-divisor graph of a ring, Comm. Algebra 36 (2008) 3073–3092.

[8]. D.F. Anderson, R. Levy, J. Shapiro, Zero-divisor graphs, von Neumann regular rings, and Boolean algebras, J. Pure Appl. Algebra 180 (2003) 221–241.

[9] D.F. Anderson, S.B. Mulay, On the diameter and girth of a zero-divisor graph, J. Pure Appl. Algebra 210 (2007) 543–550.

[10] M. Axtel, J. Coykendall, J. Stickles, Zero-divisor graphs of polynomials and power series over commutative rings, Comm. Algebra 33 (2005) 2043–2050.

[11] M. Axtel, J. Stickles, Zero-divisor graphs of idealizations, J. Pure Appl. Algebra 204 (2006) 235–243.

[12] I. Kaplansky, Commutative Rings, rev. ed., University of Chicago Press, Chicago, 1974.

[13] J.D. LaGrange, Complemented zero-divisor graphs and Boolean rings, J. Algebra 315 (2007) 600–611.

[14] T.G. Lucas, The diameter of a zero-divisor graph, J. Algebra 301 (2006) 174–193.

[15] S. Akbari, A. Mohammadian, On the zero-divisor graph of a commutative ring, J. Algebra 274 (2004) 847-855.

[16] S.E. Atani. and F.Farzalipour., Zero Divisor Graphs of idealizations with respect to modules”. Chiang Mai J.Sci.36(1),2009,5-8.